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Heating a salinity gradient from a vertical sidewall: nonlinear theory

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When a body of fluid with a vertical salinity gradient is heated from a vertical sidewall instabilities are sometimes observed. The linear stability of this basic state has been investigated by Kerr (1989). This linear theory predicts the onset of instability well when compared with experimental results; however, the form of the observed nonlinear instabilities does not coincide with the linear predictions (cf. Chen, Briggs & Wirtz 1971; Tsinober & Tanny 1986; Tanny & Tsinober 1988). In this paper we investigate some of the nonlinear aspects of the problem. A weakly nonlinear analysis reveals that the bifurcation into instability is subcritical, and that the initial trend along this branch of solutions is towards the co-rotating cells observed in experiments. The heating levels for which instabilities are absent are investigated by the use of energy stability analysis. This yields a weak result for arbitrary disturbances, showing that disturbances will decay for sufficiently low wall heating. This bound is greatly strengthened by imposing a vertical periodicity on the lengthscale proposed by Chen *et al.*

1. Introduction

When a body of fluid with a vertical salinity gradient is heated from a vertical sidewall the fluid may become unstable to thin, almost horizontal instabilities. The linear stability of a laterally heated salinity gradient was investigated in Kerr (1989, hereinafter referred to as I). This analysis uses a quasi-static approximation based on the disparity between the vertical and horizontal lengthscales of the instabilities. It was found that the linear theory predicted the onset of the instabilities well; however, the form of the instabilities from the experimental observations of Chen, Briggs & Wirtz (1971), Tsinober & Tanny (1986), and Tanny & Tsinober (1988) did not match up exactly with the form of the instabilities predicted by the linear theory. The linear theory, by its very nature, predicts that linear infinitesimal disturbances consist of a series of counter-rotating convection cells; however, in experimental observations all the convection cells circulate in the same direction, with the fluid rising at the wall before moving away into the bulk of the fluid where it gradually sinks. These instabilities are only visible once they have reached a finite amplitude, hence it is evident that nonlinear effects are important in the observations. It is the purpose of this paper to examine some of the finite-amplitude aspects of the instabilities investigated in I.

The importance of nonlinearity in observed instabilities also applies to the case of vertical and inclined slots that enclose fluid with a vertical salinity gradient when subject to lateral heating. Hart (1973) showed that a salinity gradient in a vertical slot underwent a subcritical bifurcation when heated. The finite-amplitude form of the instabilities have been investigated numerically by Wirtz, Briggs & Chen (1972) and Thangam, Zebib & Chen (1982). The former looked at both a finite vertical slot and an infinite vertical slot. In the latter case the background flow was subjected to small random disturbances to the vorticity. The instabilities that evolved consisted only of co-rotating convection cells. In the numerical investigation of Thangam *et al.* the nonlinear evolution of a vertical and sloping infinite slot was investigated. In this investigation an initial disturbance of the form of the counter-rotating convection cells predicted by linear theory was used. These soon evolved into co-rotating convection cells. In both these investigations the heating was supercritical.

In §2 a weakly nonlinear perturbation analysis is performed to find the behaviour of the instability near the critical point, using the quasi-static assumption of I. It is found that the bifurcation at the point of marginal stability is subcritical, and so instabilities whose form is close to that predicted by the linear theory of I may not be observed. This situation parallels that found in the weakly nonlinear analysis of Hart (1973) of a salinity gradient in a vertical slot subjected to lateral heating. In §3 an energy stability analysis is carried out to determine a lower bound for the degree of subcriticality of the instability. This energy stability analysis is of two parts, the first part follows the work of Dudis & Davis (1971) in their investigations of a buoyancy boundary layer near a vertical wall. In this analysis there is no restriction on the vertical scale of the instabilities and the resultant lower bound on the wall heating for the possible existence of subcritical instabilities is weak. In the second part of this energy stability analysis an extra constraint is imposed on the vertical lengthscale of the instabilities using the scale proposed by Chen *et al.* (1971). This additional constraint produces a much stronger bound on the heating rate for the possible existence of subcritical instabilities.

2. Weakly nonlinear analysis

In this section we look at the effect of nonlinearity on the instabilities observed when a semi-infinite body of fluid with a vertical salinity gradient is heated from a vertical sidewall for near marginal heating rates. It was found in I that, for strong stratification, the non-dimensional parameter that governed the stability of the fluid to infinitesimal disturbances was

$$Q = \frac{(1-\tau)^6\, g(\alpha \Delta T)^6}{\nu \kappa_S\, l^2 (-\beta \bar{S}_z)^5}, \tag{2.1}$$

where g is the acceleration due to gravity, ΔT the change of temperature at the wall, $\bar{S}_z$ the vertical salinity gradient, α the coefficient of thermal expansion, β the density change due to a unit change in the salinity, $l = (\kappa_T t)^{\frac{1}{2}}$ the horizontal lengthscale, ν the kinematic viscosity, κ_T the diffusivity of heat, κ_S the diffusivity of salt and τ the salt/heat diffusivity ratio. The second important non-dimensional parameter is δ, the ratio of the vertical lengthscale $h = (1-\tau)\,\alpha \Delta T(-\beta \bar{S}_z)^{-1}$ to the horizontal lengthscale. For the analyses to be valid in I and in this section δ must be small. There is an associated quasi-static assumption discussed in I that if δ is small then it can be considered to be a constant, independent of time. In this paper we shall only consider the error-function temperature profile associated with an instantaneous increase of the wall temperature of ΔT at time $t = 0$. We use this temperature profile as it is the large-time asymptotic limit for all changes in wall temperature that are monotonically increasing with a finite upper bound.

In this section we look at the solutions to the nonlinear governing equations when the heating rate is close to the critical value predicted by the linear theory and the amplitude of the disturbance is small. This weakly nonlinear analysis is similar in principle to that of Malkus & Veronis (1958) in their investigation of finite-amplitude convection between two horizontal surfaces. Hart (1973) also used this technique to investigate the behaviour of finite-amplitude disturbances to fluid with a vertical salinity gradient contained in a vertical slot with an imposed lateral temperature difference between the walls. Hart found that the linear instabilities underwent a subcritical bifurcation from the background state. Here we apply the weakly nonlinear analysis to the single-sidewall problem to investigate the nature of the bifurcation from the background state.

Using the non-dimensionalizations of I, the governing equations for perturbations to the background state are

$$\left(\frac{\partial}{\partial t}+\tfrac{1}{2}\delta^2-\tfrac{1}{2}\delta^2 x\frac{\partial}{\partial x}\right)\nabla_m^2 \psi+\delta^2 \bar{w}(x)\frac{\partial}{\partial z}\nabla_m^2 \psi-\delta^4\frac{\partial \psi}{\partial z}\frac{\partial^2}{\partial x^2}\bar{w}(x)$$

$$+\mathrm{J}(\psi,\nabla_m^2\psi)=\frac{\tau\sigma Q}{(1-\tau)}\left(\frac{\partial T}{\partial x}-\frac{\partial S}{\partial x}\right)+\sigma\nabla_m^4\psi, \quad (2.2a)$$

$$\left(\frac{\partial}{\partial t}-\tfrac{1}{2}\delta^2 x\frac{\partial}{\partial x}\right)T+\delta^2\bar{w}(x)\frac{\partial T}{\partial z}-\frac{\partial \psi}{\partial z}\frac{\partial f}{\partial x}+\mathrm{J}(\psi,T)=\nabla_m^2 T, \quad (2.2b)$$

$$\left(\frac{\partial}{\partial t}-\frac{\delta^2 x}{2}\frac{\partial}{\partial x}\right)S+\delta^2\bar{w}(x)\frac{\partial S}{\partial z}-\frac{\partial \psi}{\partial z}\frac{\partial f}{\partial x}-(1-\tau)\frac{\partial \psi}{\partial x}+\mathrm{J}(\psi,S)=\tau\nabla_m^2 S, \quad (2.2c)$$

where the Jacobian
$$\mathrm{J}(A,B)=\frac{\partial A}{\partial x}\frac{\partial B}{\partial z}-\frac{\partial A}{\partial z}\frac{\partial B}{\partial x}, \quad (2.2d)$$

and the Prandtl number, σ, is defined by ν/κ_T. In these equations ψ is the perturbation stream function, T and S are the perturbations to the background temperature and salinity and f and $\bar{w}$ are the background temperature profile and vertical fluid velocity.

These are the same non-dimensional governing equations as I but with the addition of the nonlinear Jacobian terms. The boundary conditions for this problem far from the wall are that all perturbations decay to zero. In I it was found that, in the limit $\delta \rightarrow 0$, thin boundary layers developed at the wall that only had an effect on the stability of the bulk of the fluid to order δ^2. This turns out to be negligible in the range of validity of this weakly nonlinear theory. The appropriate boundary conditions that are applied at the wall for the required accuracy in this theory are that the temperature perturbation T vanishes at $x=0$ and that there is no fluid flux into or out of the wall.

In this analysis we look at the solutions to these equations that have a steady amplitude that is, in some sense, small but finite, so that if ϵ is some measure of the amplitude (to be defined later) we require that $\epsilon \ll 1$. Just as in I we assume that $\delta^2 \ll 1$ in our calculations, and that in this limit δ can be taken to be a constant.

With these assumptions we look for solutions in the limits of small ϵ and δ. We expand ψ, T, S in terms of double power series in ϵ and δ. The solutions will be periodic in the vertical direction with a period $2\pi/m$, and so we can express the vertical z-dependence by splitting the solution into its Fourier modes. In this way we express each quantity as the sum of terms which have a z-dependency of the form e^{inmz} for $n \geqslant 0$. The solutions that we are looking for are of constant amplitude and form, moving upwards with a phase velocity of $-\omega/m$, and so all the Fourier modes are functions of $(mz+\omega t)$. Hence the time dependency of each of the modes will be of the form $\partial/\partial t \equiv \mathrm{i}n\omega$. We also expand Q and ω as double power series in ϵ and δ, and so

$$\psi = \sum_{j,k,n} \epsilon^j \delta^k \psi^n_{j,k}(x)\, \mathrm{e}^{\mathrm{i}n(\omega t+mz)}, \tag{2.3a}$$

$$T = \sum_{j,k,n} \epsilon^j \delta^k T^n_{j,k}(x)\, \mathrm{e}^{\mathrm{i}n(\omega t+mz)}, \tag{2.3b}$$

$$S = \sum_{j,k,n} \epsilon^j \delta^k S^n_{j,k}(x)\, \mathrm{e}^{\mathrm{i}n(\omega t+mz)}, \tag{2.3c}$$

$$Q = \sum_{j,k} \epsilon^j \delta^k Q_{j,k}, \tag{2.3d}$$

$$\omega = \sum_{j,k} \epsilon^j \delta^k \omega_{j,k}. \tag{2.3e}$$

As ϵ is a measure of the amplitude of the instabilities we have the restriction that, for ψ, T and S, $j \geqslant 1$. For Q and ω the restriction is that $j \geqslant 0$. There is no reason at this point to impose any restriction on the possible powers of δ in these expansions. As we are interested only in the real parts of the solutions we have the relation that

$$\mathrm{Re}\,\{\psi(x)\,\mathrm{e}^{\mathrm{i}n(mz+\omega t)}\} = \mathrm{Re}\,\{\bar{\psi}(x)\,\mathrm{e}^{-\mathrm{i}n(mz+\omega t)}\}, \tag{2.4}$$

and hence restrict ourselves to considering the case where $n \geqslant 0$.

We substitute these expansions into (2.1) and look at the $O(\epsilon)$ terms. At this order the nonlinear terms do not appear. The resultant equations for the $n=1$ mode are

$$-\mathrm{i}\omega_{0,0} m^2 \psi^1_{1,0} = \frac{\sigma\tau}{(1-\tau)} Q_{0,0}\left(\frac{\mathrm{d}T^1_{1,0}}{\mathrm{d}x} - \frac{\mathrm{d}S^1_{1,0}}{\mathrm{d}x}\right) + \sigma m^4 \psi^1_{1,0} + O(\delta^2), \tag{2.5a}$$

$$\mathrm{i}\omega_{0,0} T^1_{1,0} - \mathrm{i}m\psi^1_{1,0} f'(x) = -m^2 T^1_{1,0} + O(\delta^2), \tag{2.5b}$$

$$\mathrm{i}\omega_{0,0} S^1_{1,0} - \mathrm{i}m\psi^1_{1,0} f'(x) - (1-\tau)\frac{\mathrm{d}\psi^1_{1,0}}{\mathrm{d}x} = -\tau m^2 S^1_{1,0} + O(\delta^2). \tag{2.5c}$$

This is the same set of equations as found in the linear analysis of I, but this time more care must be taken in finding the solution since we are now concerned with the amplitude. As these equations are both linear and homogeneous, a solution when multiplied by any constant is also a solution. We still have the freedom to choose any measure of the amplitude of the instabilities, but different choices will give different values of coefficients associated with the nonlinear motions. We make the choice of amplitude so that the order-ϵ solution satisfies

$$\int_0^\infty \bar{\psi}^1_{1,0}(x)\,\psi^1_{1,0}(x)\,\mathrm{d}x = 1. \tag{2.6}$$

This does not totally remove the degeneracy since any solution when multiplied by a complex number with magnitude 1 will also satisfy this criterion. This degeneracy is removed by requiring that the derivative of $\psi^1_{1,0}$ is real and positive at $x=0$. We now define the amplitude, ϵ, of any finite-amplitude solution, $\psi(x,z,t)$, by

$$\epsilon = \left|\frac{m}{2\pi}\int_0^{2\pi/m}\int_0^\infty \bar{\psi}^1_{1,0}(x)\,\mathrm{e}^{-\mathrm{i}mz}\,\psi(x,z,t)\,\mathrm{d}x\,\mathrm{d}z\right|. \tag{2.7}$$

We can also define the phase, γ, of the instability by

$$\epsilon e^{i\gamma} = \frac{m}{2\pi}\int_0^{2\pi/m}\int_0^{\infty} \bar{\psi}^1_{1,0}(x)\, e^{-imz}\, \psi(x,z,t)\,dx\,dz. \tag{2.8}$$

In particular, if we substitute in the expansion (2.3a) for ψ we get

$$\left|\sum_{j,k}\int_0^{\infty} \bar{\psi}^1_{1,0}(x)\,\epsilon^j\,\delta^k\,\psi^1_{j,k}(x)\,dx\right| = \epsilon. \tag{2.9}$$

From this we deduce that all higher-order modes with $n = 1$ are orthogonal to the first mode, $\psi^1_{1,0}$.

The solutions to (2.5) have been found in I, where the marginally stable solution was found. The leading-order solution for $\sigma = 7$ and $\tau = 1/80$, the approximate values for common salt in water, corresponds to the critical solution found there:

$$\left.\begin{aligned} \psi^1_{1,0} = \psi_{\text{crit}},\quad T^1_{1,0} = T_{\text{crit}},\quad S^1_{1,0} = S_{\text{crit}},\\ Q_{0,0} = Q_{\text{crit}} = 147\,700,\quad \omega_{0,0} = \omega_{\text{crit}} = 0.6744.\end{aligned}\right\} \tag{2.10}$$

This solution occurs for the critical value of $m = 6.244$. Here ψ_{crit}, T_{crit} and S_{crit} are the corresponding stream function, temperature perturbation and salinity perturbation renormalized to have unit amplitude. With this value of $Q_{0,0}$ all other linear modes with different values of m are stable. Since there are no forcing terms at this order for any Fourier mode with $n \neq 1$, this is the only possible non-zero solution.

We now look at the $O(\epsilon^2)$ terms. At this order we get the first influence of the nonlinear Jacobian terms. These Jacobian terms force a response in the order-ϵ^2 equations for both the $n = 0$ and $n = 2$ modes. There are also forcing terms for the $n = 1$ mode that come from the $O(\epsilon)$ terms in the expansions of Q and ω. All other modes are stable. We split the $O(\epsilon^2)$ equations into these three parts and solve them separately.

First we consider the terms with the same vertical periodicity as the first-order equations. Using the notation that the absence of the second lower suffix indicates that the δ-dependency has not yet been determined, the leading-order equations for this mode are

$$(i\omega_{0,0}+\sigma m^2)\,m^2\,\psi^1_2 + \frac{\sigma\tau}{(1-\tau)}Q_{0,0}\left(\frac{dT^1_2}{dx}-\frac{dS^1_2}{dx}\right) = -i\omega_1\,\psi^1_{1,0} - \frac{\sigma\tau}{(1-\tau)}Q_1\left(\frac{dT^1_{1,0}}{dx}-\frac{dS^1_{1,0}}{dx}\right), \tag{2.11a}$$

$$(i\omega_{0,0}+m^2)\,T^1_2 - im\psi^1_2 f'(x) = -i\omega_1\,T^1_{1,0}, \tag{2.11b}$$

$$(i\omega_{0,0}+\tau m^2)\,S^1_2 - im\psi^1_2 f'(x) - (1-\tau)\frac{d\psi^1_2}{dx} = -i\omega_1\,S^1_{1,0}. \tag{2.11c}$$

In I a solvability condition was derived for sets of equations of this form in order for a solution to exist. In I the boundary condition for ψ at $x = 0$ was that ψ took an $O(\delta^2)$ value. Here it is sufficient to set $\psi(0) = 0$, and so the solvability condition obtained by multiplying (2.11) by the respective conjugates of the adjoints $\hat{\psi}$, $\hat{T}$ and $\hat{S}$ and integrating the sum of the resultant equations from 0 to ∞ is

$$i\omega_1\int_0^{\infty} m^2\bar{\hat{\psi}}\psi^1_{1,0} + \bar{\hat{T}}T^1_{1,0} + \bar{\hat{S}}S^1_{1,0}\,dx + \frac{\sigma\tau}{(1-\tau)}Q_1\int_0^{\infty}\bar{\hat{\psi}}\left(\frac{dT^1_{1,0}}{dx}-\frac{dS^1_{1,0}}{dx}\right)dx = 0. \tag{2.12}$$

Since both Q_1 and ω_1 are real we can take the real and imaginary parts of this equation to give a pair of simultaneous equations for the two unknowns, obtaining the trivial solution

$$\omega_1 = Q_1 = 0. \tag{2.13}$$

This result is true for all orders of δ. This result is to be expected, otherwise the nonlinear behaviour would depend on the sign of the amplitude, not just its magnitude. However, changing the sign of the solutions is equivalent to a vertical phase shift of the disturbances by half a period, and should have no effect on their behaviour.

Since there is now no forcing to these equations, the only possible solutions are multiples of the critical solution found for the first-order equations. However, the orthogonality relation between $\psi^1_{1,0}$ and all the higher-order modes means that $\psi^1_{2,0}$, $T^1_{2,0}$ and $S^1_{2,0}$ must all be zero.

Next we look at the leading-order equations for the terms proportional to exp $(2\mathrm{i}(mz+\omega t))$:

$$-(2\mathrm{i}\omega_{0,0}+4\sigma m^2)\,4m^2\psi^2_{2,0}-\frac{\sigma\tau}{(1-\tau)}Q_{0,0}\left(\frac{\mathrm{d}T^2_{2,0}}{\mathrm{d}x}-\frac{\mathrm{d}S^2_{2,0}}{\mathrm{d}x}\right)=0, \tag{2.14a}$$

$$(2\mathrm{i}\omega_{0,0}+4m^2)\,T^2_{2,0}-2\mathrm{i}m\psi^2_{2,0}f'(x)=-\tfrac{1}{2}\mathrm{i}m\left(T^1_{1,0}\frac{\mathrm{d}\psi^1_{1,0}}{\mathrm{d}x}-\psi^1_{1,0}\frac{\mathrm{d}T^1_{1,0}}{\mathrm{d}x}\right), \tag{2.14b}$$

$$(2\mathrm{i}\omega_{0,0}+4\tau m^2)\,S^2_{2,0}-2\mathrm{i}m\psi^2_{2,0}f'(x)-(1-\tau)\frac{\mathrm{d}\psi^2_{2,0}}{\mathrm{d}x}=-\tfrac{1}{2}\mathrm{i}m\left(S^1_{1,0}\frac{\mathrm{d}\psi^1_{1,0}}{\mathrm{d}x}-\psi^1_{1,0}\frac{\mathrm{d}S^1_{1,0}}{\mathrm{d}x}\right). \tag{2.14c}$$

Here the right-hand side represents the forcing from the nonlinear Jacobian terms. These equations can be solved numerically in a similar fashion to the $O(\epsilon)$ equations to find $\psi^2_{2,0}$, $T^2_{2,0}$ and $S^2_{2,0}$.

Lastly we come to the z- and t-independent terms. In the governing equations for this mode the $\partial/\partial z$ and $\partial/\partial t$ terms all vanish and so the terms with a δ^2 dependency appear at leading order:

$$\delta^4\left(\tfrac{1}{2}-\tfrac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x}\right)\frac{\mathrm{d}^2\psi^0_2}{\mathrm{d}x^2}-\frac{\sigma\tau}{(1-\tau)}Q_{0,0}\left(\frac{\mathrm{d}T^0_2}{\mathrm{d}x}-\frac{\mathrm{d}S^0_2}{\mathrm{d}x}\right)-\sigma\delta^4\frac{\mathrm{d}^4}{\mathrm{d}x^4}\psi^0_2$$
$$=-\tfrac{1}{2}\mathrm{i}m\delta^2\left(\frac{\mathrm{d}\bar{\psi}^1_{1,0}}{\mathrm{d}x}\frac{\mathrm{d}^2\psi^1_{1,0}}{\mathrm{d}x^2}+\bar{\psi}^1_{1,0}\frac{\mathrm{d}^3\psi^1_{1,0}}{\mathrm{d}x^3}\right), \tag{2.15a}$$

$$-\tfrac{1}{2}\delta^2x\frac{\mathrm{d}T^0_2}{\mathrm{d}x}-\delta^2\frac{\mathrm{d}^2T^0_2}{\mathrm{d}x^2}=-\tfrac{1}{2}\mathrm{i}m\frac{\mathrm{d}}{\mathrm{d}x}(\bar{\psi}^1_{1,0}T^1_{1,0}), \tag{2.15b}$$

$$-\tfrac{1}{2}\delta^2x\frac{\mathrm{d}S^0_2}{\mathrm{d}x}-\delta^2\tau\frac{\mathrm{d}^2S^0_2}{\mathrm{d}x^2}-(1-\tau)\frac{\mathrm{d}\psi^0_2}{\mathrm{d}x}=-\tfrac{1}{2}\mathrm{i}m\frac{\mathrm{d}}{\mathrm{d}x}(\bar{\psi}^1_{1,0}S^1_{1,0}). \tag{2.15c}$$

Equation (2.15*b*) has the solution

$$T^0_{2,-2}=-\tfrac{1}{2}\mathrm{i}m\int_x^\infty \mathrm{e}^{-\frac{1}{4}p^2}\int_0^p \mathrm{e}^{\frac{1}{4}q^2}\frac{\mathrm{d}}{\mathrm{d}q}(\bar{\psi}^1_{1,0}T^1_{1,0})\,\mathrm{d}q\,\mathrm{d}p$$
$$+\tfrac{1}{2}\mathrm{i}m\,\pi^{-\frac{1}{2}}\int_0^\infty \mathrm{e}^{-\frac{1}{4}q^2}\int_0^q \mathrm{e}^{\frac{1}{4}r^2}\frac{\mathrm{d}}{\mathrm{d}r}(\bar{\psi}^1_{1,0}T^1_{1,0})\,\mathrm{d}r\,\mathrm{d}q\int_x^\infty \mathrm{e}^{-\frac{1}{4}p^2}\,\mathrm{d}p. \tag{2.16}$$

Note that this term is of order $\epsilon^2\delta^{-2}$. In all the other modes, with $n \neq 0$, the forcing terms are balanced by vertical diffusion over a lengthscale of the convection cell thickness. However, this mode is uniform in the vertical direction and so there is no vertical diffusion. The only effects that can balance the forcing terms in this mode are the weak horizontal diffusion and advection, both of order δ^2 weaker than the vertical diffusion for modes with $n \neq 0$. Since these limiting effects are weak the perturbation has to grow to a correspondingly larger amplitude, a factor of order δ^{-2} larger than would otherwise be the case, before a balance occurs between the driving terms in the equations and the balancing effects.

From (2.15*a*) we find that, unless ψ_2^0 is of order δ^{-6}, at leading order the horizontal salinity and temperature gradients will balance. As both perturbations decay as $x \rightarrow \infty$ this gives for the bulk of the fluid

$$T^0_{2,-2} = S^0_{2,-2}. \tag{2.17}$$

The salinity does not match up with the appropriate physical salt boundary condition and this results in a boundary layer forming with thickness proportional to $\delta^{\frac{3}{2}}$ (see §5 of I). As mentioned earlier, the effect of this boundary layer is negligible in this analysis.

The last of the three equations, (2.15*c*), gives the z-independent perturbation to the stream function:

$$\psi^0_{2,0} = \frac{1}{(1-\tau)}\left\{\tfrac{1}{2}\mathrm{i}m\,\bar{\psi}^1_{1,0}S^1_{1,0} - \tfrac{1}{2}xS^0_{2,-2} + \tau\frac{\mathrm{d}S^0_{2,-2}}{\mathrm{d}x} - \frac{1}{2}\int_x^\infty S^0_{2,-2}(x')\,\mathrm{d}x'\right\}. \tag{2.18}$$

This leading-order z-independent part of the stream function is of order ϵ^2 and not of order $\epsilon^2\delta^{-2}$.

We now look at the $O(\epsilon^3)$ equations. This time the Jacobian terms give a forcing to the $n = 1$ mode and the $n = 3$ mode. The equations for the terms proportional to $\exp(\mathrm{i}(mz+\omega t))$ have forcing terms of order $\epsilon^2\delta^{-2}$ for both the salinity and temperature parts. The leading-order equations are

$$(\mathrm{i}\omega_{0,0}+\sigma m^2)m^2\psi^1_{3,-2} + \frac{\sigma\tau}{(1-\tau)}Q_{0,0}\left(\frac{\mathrm{d}T^1_{3,-2}}{\mathrm{d}x} - \frac{\mathrm{d}S^1_{3,-2}}{\mathrm{d}x}\right)$$
$$= -\mathrm{i}\omega_{2,-2}m^2\psi^1_{1,0} - \frac{\sigma\tau}{(1-\tau)}Q_{2,-2}\left(\frac{\mathrm{d}T^1_{1,0}}{\mathrm{d}x} - \frac{\mathrm{d}S^1_{1,0}}{\mathrm{d}x}\right), \tag{2.19a}$$

$$(\mathrm{i}\omega_{0,0}+m^2)T^1_{3,-2} - \mathrm{i}m\psi^1_{3,-2}f'(x) = -\mathrm{i}\omega_{2,-2}T^1_{1,0} + \mathrm{i}m\psi^1_{1,0}\frac{\mathrm{d}T^0_{2,-2}}{\mathrm{d}x}, \tag{2.19b}$$

$$(\mathrm{i}\omega_{0,0}+\tau m^2)S^1_{3,-2} - \mathrm{i}m\psi^1_{3,-2}f'(x) - (1-\tau)\frac{\mathrm{d}\psi^1_{3,-2}}{\mathrm{d}x} = -\mathrm{i}\omega_{2,-2}S^1_{1,0} + \mathrm{i}m\psi^1_{1,0}\frac{\mathrm{d}S^0_{2,-2}}{\mathrm{d}x}. \tag{2.19c}$$

We apply the solvability condition to these equations and obtain the criterion for the existence of a solution to these equations:

$$0 = \mathrm{i}\omega_{2,-2}\int_0^\infty m^2\bar{\tilde{\psi}}\psi^1_{1,0} + \bar{\tilde{T}}T^1_{1,0} + \bar{\tilde{S}}S^1_{1,0}\,\mathrm{d}x$$
$$+ \frac{\sigma\tau}{(1-\tau)}Q_{2,-2}\int_0^\infty \bar{\tilde{\psi}}\left(\frac{\mathrm{d}T^1_{1,0}}{\mathrm{d}x} - \frac{\mathrm{d}S^1_{1,0}}{\mathrm{d}x}\right)\mathrm{d}x$$
$$- \int_0^\infty \mathrm{i}m\bar{\tilde{T}}\psi^1_{1,0}\frac{\mathrm{d}T^0_{2,-2}}{\mathrm{d}x} + \mathrm{i}m\bar{\tilde{S}}\psi^1_{1,0}\frac{\mathrm{d}S^0_{2,-2}}{\mathrm{d}x}\,\mathrm{d}x. \tag{2.20}$$

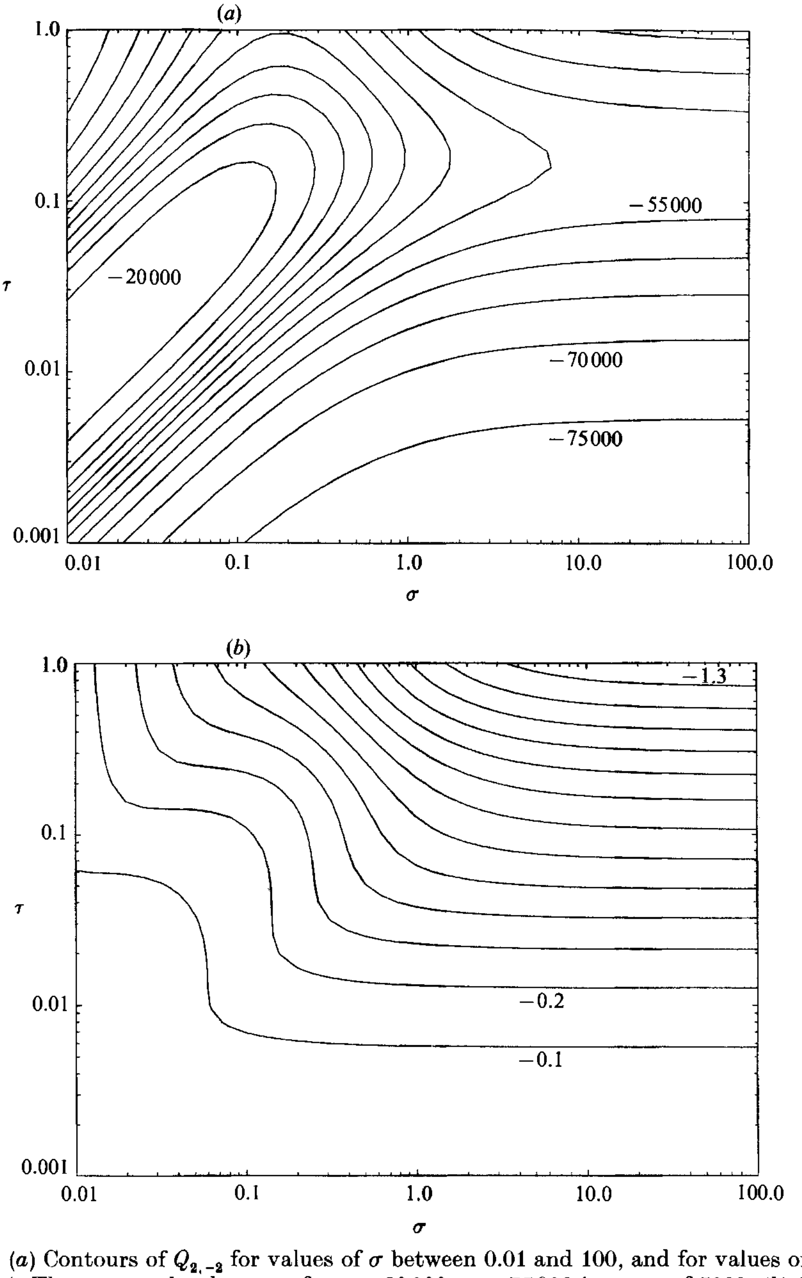

FIGURE 1. (*a*) Contours of $Q_{2,-2}$ for values of σ between 0.01 and 100, and for values of τ between 0.001 and 1. The contour levels range from -20000 to -75000 in steps of 5000. (*b*) Contours of $\omega_{2,-2}$ for values of σ between 0.01 and 100, and for values of τ between 0.001 and 1. The contour levels range from -0.1 to -1.3 in steps of 0.1.

We take the real and imaginary parts of this equation and solve the resultant simultaneous equations. Unlike the previous order this yields non-zero values for the perturbations to Q and ω, the calculated values of which for $\sigma = 7$ and $\tau = 1/80$ are

$$Q_{2,-2} = -71{,}010, \quad \omega_{2,-2} = -0.1967. \qquad (2.21a, b)$$

The values of $Q_{2,-2}$ and $\omega_{2,-2}$ for values of σ between 0.01 and 100 and for τ between

0.001 and 1 are shown in figure 1. The contours of constant $Q^{2,-2}$ and $\omega_{2,-2}$ both show a symmetry about the line $\sigma = \tau$. It was found in I that, owing to the symmetry of the differential equation for ψ to the interchange of σ and τ, $Q_{0,0}$, $\omega_{0,0}$ and $\psi_{0,0}$ are unchanged if the values of σ and τ are interchanged. This is also the case for $\hat{\psi}$. If (2.20) is re-expressed in terms of ψ and $\hat{\psi}$ and multiplied by a factor of $(\mathrm{i}\omega_{0,0}+\tau m^2)$ then this equation can also be arranged to reflect this σ/τ symmetry, and hence $Q_{2,-2}$ and $\omega^{2,-2}$ are also unchanged if the values of σ and τ are interchanged. The physical significance of this symmetry is not understood.

We now have expressions for Q and ω of the form

$$Q = Q_{0,0}+\epsilon^2\delta^{-2}Q_{2,-2}\dots, \tag{2.22a}$$

and

$$\omega = \omega_{0,0}+\epsilon^2\delta^{-2}\omega_{2,-2}+\dots. \tag{2.22b}$$

To derive these expression we have assumed that both ϵ and δ are small, but this does not tell us anything about the size of $\epsilon^2\delta^{-2}$. Examining how the various terms that give rise to the negative powers of δ interact tells us that each factor of δ^{-2} is associated with a factor of ϵ^2. For the resulting expansion to give an asymptotic series we must require that

$$\epsilon^2\delta^{-2} \ll 1. \tag{2.23}$$

From the quasi-static assumption of I we have the requirement that the value of Q should differ from the critical value by an amount much greater than $O(\delta^2)$, hence

$$Q-Q_{\mathrm{crit}} = O(\epsilon^2\delta^{-2}) \gg \delta^2, \tag{2.24}$$

and so we require

$$\epsilon^2 \gg \delta^4. \tag{2.25}$$

These two requirements can be combined to give the condition for this weakly nonlinear asymptotic analysis to be valid that

$$\delta^4 \ll \epsilon^2 \ll \delta^2. \tag{2.26}$$

From this we can see that for a given small δ there is a range of values of ϵ for which the asymptotic expansion is valid.

The values of the perturbation to Q are negative for all the values of σ and τ investigated. This implies that we have found solutions to the nonlinear equations that exist for values of Q less than the critical value of Q found by the linear analysis of I. Since the system is stable to infinitesimal disturbances for values of Q less than that of the bifurcation point Q_{crit}, and unstable for larger values of Q, the subcritical branch of solutions that has been found will itself be unstable. Hence in reality these steady, finite-amplitude solutions near the bifurcation point would not be observed. If it were possible to observe them for values of Q close to the Q_{crit} then small-amplitude solutions would exist that, to a good approximation, looked like the linear solutions. However, the linear results have counter-rotating convection cells, and experiments have only revealed cells that all rotate in the same direction. These observations suggest that any branch of steady, finite-amplitude solutions near the bifurcation point should indeed be unstable. However, this branch of unstable solutions may become stable for larger amplitudes than the weakly nonlinear analysis can access. If this is the case then the form of the deviation of the solutions from the linear solution of I may give an indication of what would be observed in a large-amplitude disturbance.

The perturbations to both the temperature and the salinity are dominated by the z-independent terms. In the regime for which this analysis is valid these perturbations are larger in magnitude than the first-order convection cells that cause them. A plot

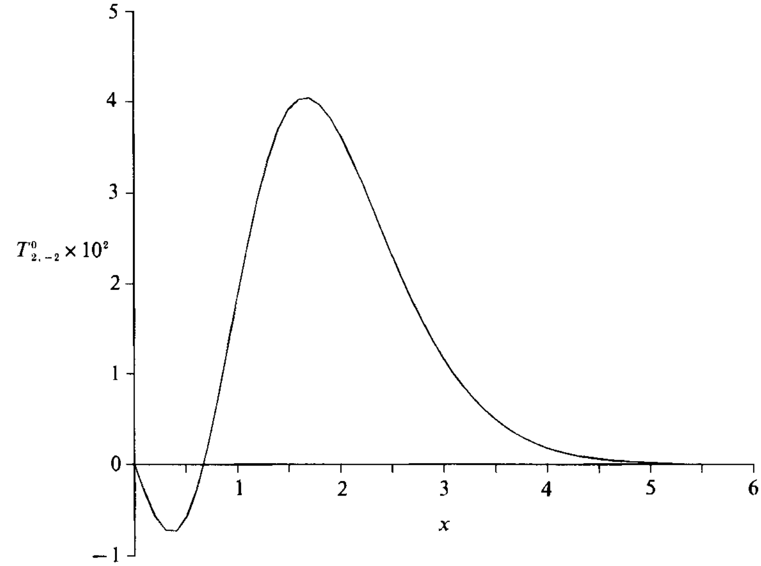

FIGURE 2. Graph of the leading-order z-independent temperature perturbation, $T^0_{2,-2}$, as a function of the distance from the wall for the case $\sigma = 7$ and $\tau = 1/80$.

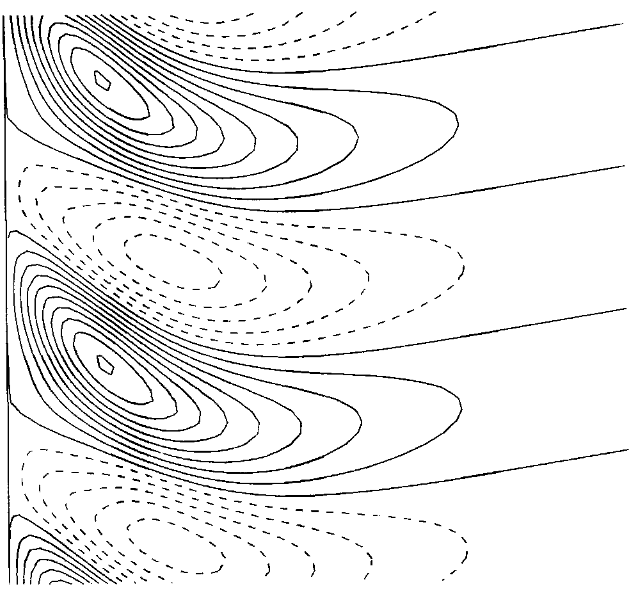

FIGURE 3. The streamlines for a weakly nonlinear disturbance near to the critical value of Q; here $\epsilon^2 = 0.1$, $\sigma = 7$ and $\tau = 1/80$. These show that the convection cells with fluid rising near the wall are enhanced, while the other cells are diminished. This solution is unstable and would not be observed.

of $T^0_{2,-2}$ against x is shown in figure 2. There is a decrease in the temperature near the wall and an increase further away. This is to be expected since any convection would be expected to increase the flux of heat away from the wall, thus heating up the distant fluid more than would be the case for the purely conducting background state. This enhanced heat flux caused by the convection will require a greater flux of heat from the wall itself. At the wall the flux is due only to conduction and so there

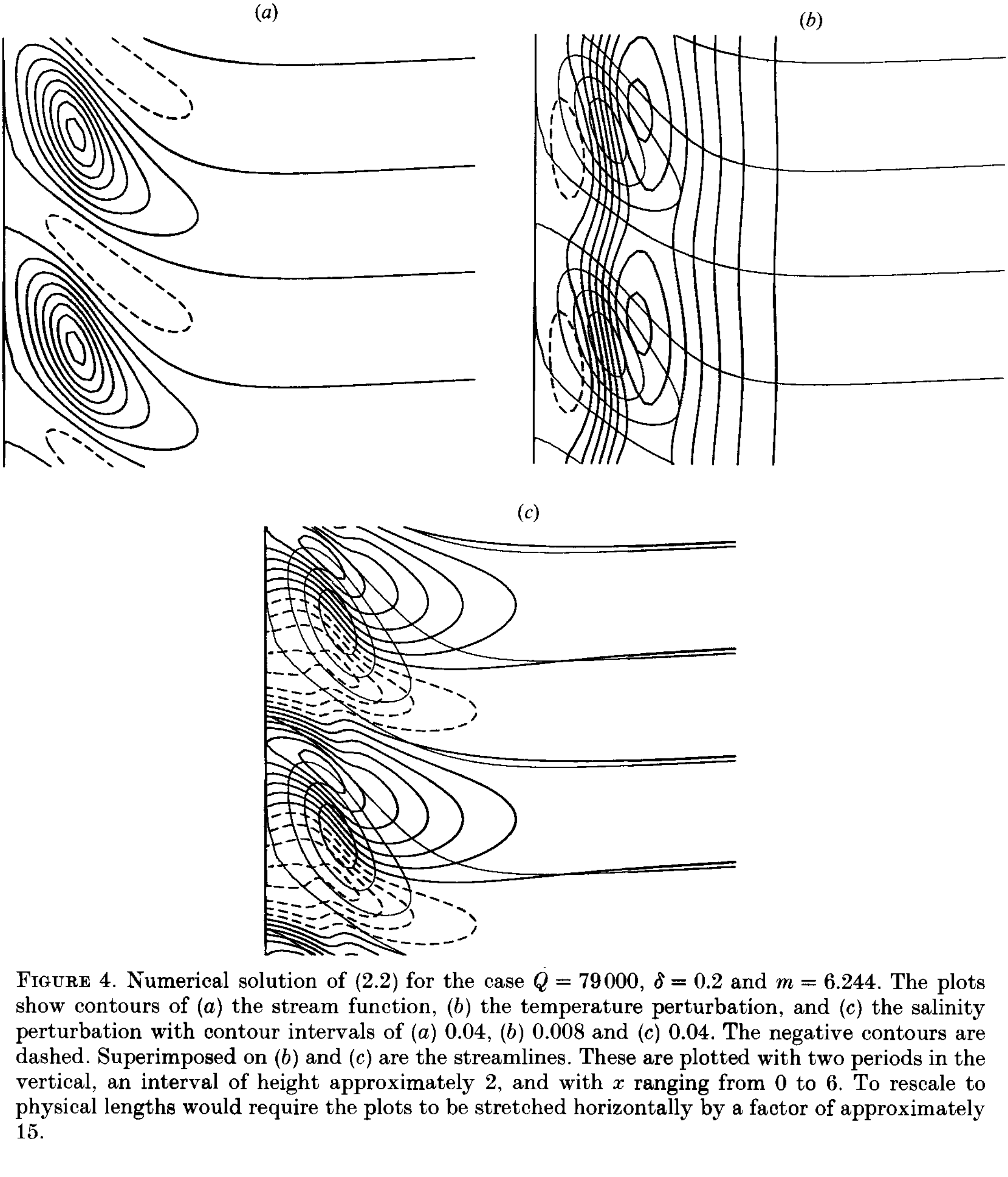

FIGURE 4. Numerical solution of (2.2) for the case $Q = 79000$, $\delta = 0.2$ and $m = 6.244$. The plots show contours of (*a*) the stream function, (*b*) the temperature perturbation, and (*c*) the salinity perturbation with contour intervals of (*a*) 0.04, (*b*) 0.008 and (*c*) 0.04. The negative contours are dashed. Superimposed on (*b*) and (*c*) are the streamlines. These are plotted with two periods in the vertical, an interval of height approximately 2, and with x ranging from 0 to 6. To rescale to physical lengths would require the plots to be stretched horizontally by a factor of approximately 15.

must be an enhanced temperature gradient there. This requires the presence of an area with a negative temperature perturbation close to the wall.

The z-independent perturbation to the stream function is positive in the region near the wall where the convection cells are strongest. Hence the convection cells that have the fluid rising near the wall are enhanced, while the others are diminished (see figure 3). If this trend continues for the larger-amplitude solutions then the cells with fluid rising near the wall may eventually swallow up the counter-rotating cells. This would fit in with the experimental observations where only cells with fluid rising near the wall are observed.

The value of $\omega_{2,-2}$ is negative, and so the phase velocity of the convection cells decreases as their amplitude increases. This trend, if continued for large-amplitude solutions to the full equations, may result in the cells slowing down significantly. This could account for the lack of any reported observation of vertical movement in the experimental disturbances.

Although this analysis cannot tell us about large-scale disturbances to the background state, it does give indications of the form that they might take. To find large-amplitude solutions it is necessary to solve the complete set of equations (2.2). This is outside the scope of this paper, although the numerical results of Kerr (1987) do support the trends postulated above for fully developed disturbances. A typical numerical result is shown in figure 4. This result is obtained using a numerical scheme that expresses the solutions as Fourier series in the vertical, with an imposed periodicity, and uses a finite-difference scheme for the x-dependency. This example was calculated for $\sigma = 7$, $\tau = 1/80$, $m = 6.244$ and $\delta = 0.2$. The heating in this example was subcritical, using a value of Q of 79000, around half the critical value for infinitesimal disturbances. In this solution the angular frequency of the instabilities was $\omega = 0.137$, a value that is only about 22% of the critical value calculated in the linear analysis of I. In this solution the convection cells, figure 4(a), that have the fluid rising at the wall are much larger than the counter-rotating cells, which have almost vanished. The temperature perturbation, figure 4(b), shows that the dominant feature is a negative perturbation near the wall, with a corresponding rise in the mean temperature perturbation further away from the wall. This temperature perturbation had a relatively weak vertical variation, as may be anticipated from the weakly nonlinear analysis. The salinity perturbation, figure 4(c), has a stronger periodic element and for this value of δ the z-independent variation does not dominate. All these features conform with the indications from the weakly nonlinear analysis. Further details can be found in Kerr (1987).

3. Energy stability analysis

In the previous section we demonstrated that the bifurcation from the stable solution is subcritical and so finite-amplitude solutions can exist for values of Q less than the critical value found for infinitesimal disturbances in I. In this section we use energy stability theory (cf. Joseph 1976a, b) to examine the stability of the background state to finite-amplitude disturbances. This method enables us to find a lower bound for the value of Q below which disturbances in some sense die away. The first part of this analysis follows a similar course to the analysis of the steady thermal boundary layer that occurs when a vertical temperature gradient is heated at a single vertical sidewall (Dudis & Davis 1971; and Joseph 1976b, pp. 29–33). We take the full equations for perturbations to the background flow and non-dimensionalize them with respect to the following quantities:

$$T \quad \text{with respect to} \quad \Delta T, \tag{3.1a}$$

$$S \quad \text{with respect to} \quad \alpha \Delta T/\beta, \tag{3.1b}$$

$$\boldsymbol{x} \quad \text{with respect to} \quad h^* = \alpha \Delta T/(-\beta \bar{S}_z), \tag{3.1c}$$

$$t \quad \text{with respect to} \quad h^{*2}/\kappa_T, \tag{3.1d}$$

$$\boldsymbol{u} \quad \text{with respect to} \quad \kappa_T/h^*, \tag{3.1e}$$

$$p \quad \text{with respect to} \quad \rho_0 \kappa_T^2/h^{*2}. \tag{3.1f}$$

These non-dimensionalizations differ from those used in I and §2. The resulting non-dimensional equations are

$$\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\bar{\boldsymbol{U}}\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\bar{\boldsymbol{U}}=-\nabla p+\mathscr{R}(T-S)\hat{\boldsymbol{z}}+\sigma\nabla^2\boldsymbol{u}, \tag{3.2a}$$

$$\frac{\partial T}{\partial t}+\boldsymbol{u}\cdot\nabla T+\bar{\boldsymbol{U}}\cdot\nabla T+\boldsymbol{u}\cdot\nabla\bar{T}=\nabla^2 T, \tag{3.2b}$$

$$\frac{\partial S}{\partial t}+\boldsymbol{u}\cdot\nabla S+\bar{\boldsymbol{U}}\cdot\nabla S+\boldsymbol{u}\cdot\nabla\bar{S}=\tau\nabla^2 S, \tag{3.2c}$$

$$\nabla\cdot\boldsymbol{u}=0, \tag{3.2d}$$

where the background state is given by

$$\bar{\boldsymbol{U}}=(0,0,\bar{W}(w,t)), \tag{3.3a}$$

$$\bar{T}=f(x,t), \tag{3.3b}$$

$$\bar{S}=f(x,t)-z. \tag{3.3c}$$

The Prandtl number, σ, and the salt/heat diffusivity ratio, τ, are defined as before. The new non-dimensional number to appear here is

$$\mathscr{R}=\frac{g\alpha\Delta T h^3}{\kappa_T^2}. \tag{3.4}$$

The boundary conditions imposed on the perturbations are

$$\boldsymbol{u}=\boldsymbol{0},\quad T=0,\quad \frac{\partial S}{\partial x}=0\quad \text{at}\, x=0, \tag{3.5a}$$

and

$$\boldsymbol{u}\to\boldsymbol{0},\quad T\to 0,\quad S\to 0\qquad \text{as}\, x\to\infty. \tag{3.5b}$$

In this analysis the perturbations to the background state are restricted to those whose maximum amplitudes are bounded in the whole fluid region and which are absolutely integrable on $0\leqslant x<\infty$. We define the average of a quantity over y and z by

$$\bar{A}(x,t)=\lim_{K,L\to\infty}\frac{1}{4KL}\int_{-L}^{+L}\int_{-K}^{+K}A(x,y,z,t)\,\mathrm{d}y\,\mathrm{d}z \tag{3.6a}$$

and the brackets $\langle\ \rangle$ by

$$\langle A\rangle=\int_0^\infty \bar{A}(x,t)\,\mathrm{d}x. \tag{3.6b}$$

Taking products of (3.3*a–c*) with $\boldsymbol{u}$, T and S as appropriate and finding their averages we obtain, after and substituting for $\bar{\boldsymbol{U}}$, $\bar{T}$ and $\bar{S}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tfrac{1}{2}|\boldsymbol{u}|^2\rangle+\left\langle wu\frac{\partial}{\partial x}\bar{W}(x,t)\right\rangle=\mathscr{R}\langle Tw-Sw\rangle-\sigma\langle|\nabla\boldsymbol{u}|^2\rangle, \tag{3.7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tfrac{1}{2}T^2\rangle+\left\langle Tu\frac{\partial}{\partial x}f(x,t)\right\rangle=-\langle|\nabla T|^2\rangle, \tag{3.7b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tfrac{1}{2}S^2\rangle+\left\langle Su\frac{\partial}{\partial x}f(x,t)\right\rangle-\langle Sw\rangle=-\tau\langle|\nabla S|^2\rangle, \tag{3.7c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle TS\rangle+\left\langle (T+S)\,u\frac{\partial}{\partial x}f(x,t)\right\rangle-\langle Tw\rangle=-(1+\tau)\,\langle\nabla T\cdot\nabla S\rangle. \tag{3.7d}$$

Taking the sum of these, with (3.7*b*–*d*) multiplied by weightings λ_b, λ_c and λ_d respectively, we obtain

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}=\mathscr{I}-\mathscr{D} \tag{3.8a}$$

where

$$\mathscr{E}=\langle\tfrac{1}{2}|\boldsymbol{u}|^2+\lambda_b\tfrac{1}{2}T^2+\lambda_c\tfrac{1}{2}S^2+\lambda_d TS\rangle, \tag{3.8b}$$

$$\begin{aligned}\mathscr{I}=&-\left\langle wu\frac{\partial}{\partial x}\overline{W}(x,t)\right\rangle+\mathscr{R}\langle Tw-Sw\rangle-\lambda_b\left\langle Tu\frac{\partial}{\partial x}f(x,t)\right\rangle\\&-\lambda_c\left\langle Su\frac{\partial}{\partial x}f(x,t)\right\rangle+\lambda_c\langle Sw\rangle-\lambda_d\left\langle (T+S)\,u\frac{\partial}{\partial x}f(x,t)\right\rangle+\lambda_d\langle Tw\rangle,\end{aligned} \tag{3.8c}$$

$$\mathscr{D}=\langle\sigma|\nabla\boldsymbol{u}|^2+\lambda_b|\nabla T|^2+\lambda_c\tau|\nabla S|^2+\lambda_d(1+\tau)\,\nabla T\cdot\nabla S\rangle. \tag{3.8d}$$

The values of λ_b, λ_c and λ_d are chosen so that both $\mathscr{E}$ and $\mathscr{D}$ are always positive definite.

If we have the condition, over the set of all admissible functions $\boldsymbol{u}$, T and S, that

$$\sup\left\{\frac{\mathscr{I}}{\mathscr{D}}\right\}\leqslant A<1, \tag{3.9}$$

where A is some constant, then

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t}\leqslant-(1-A)\mathscr{D}. \tag{3.10}$$

As Dudis & Davis (1971) demonstrated, since the fluid region is unbounded there is no relationship of the form

$$\sup\left\{\frac{\langle\psi^2\rangle}{\langle|\nabla\psi|^2\rangle}\right\}\leqslant B<\infty, \tag{3.11}$$

or its vector equivalent. Since this supremum is infinite, even if (3.10) holds we cannot show that $\mathscr{E}\to 0$ as $t\to\infty$. However, they demonstrated that the disturbance vorticity will decay to 0 for large time. Dudis & Davis then went on to show that, although this does not imply that $\mathscr{E}\to 0$, the energy of the disturbance contained between the wall and some arbitrary fixed distance from the wall will decay to 0, and so the energy of a perturbation is dispersed over an ever increasing volume.

The analysis of Dudis & Davis can be adapted to the situation under consideration here to provide a bound of the form (3.9). For the error-function temperature profile and the associated large-time salinity profile and upwelling, the background state can be shown to be stable to arbitrary disturbances if

$$\sup\left\{\frac{\mathscr{I}}{\mathscr{D}}\right\}\leqslant\frac{(1-\tau)\,\delta^*}{4\sigma\pi^{\frac{1}{2}}}2.2283+\frac{\mathscr{R}^{\frac{1}{2}}(1-\tau)}{2\delta^*\pi^{\frac{1}{2}}(\sigma\tau)^{\frac{1}{2}}}1.4903<1, \tag{3.12}$$

where

$$\delta^*=h^*/l \tag{3.13}$$

is the ratio of the lengthscale h^* to the thermal diffusion lengthscale, $l=(\kappa_T t)^{\frac{1}{2}}$. The details of this analysis are presented in the document held in the *Journal of Fluid Mechanics* office and obtainable from there on request. Expressing this in terms of Q and the original definition of δ used in I and §2 we obtain the condition, for small δ, that the fluid is stable to all disturbances of arbitrary amplitude if

$$Q < \frac{4\pi\delta^4}{(1-\tau)^2}\frac{1}{(1.4903)^2}\left(1-\frac{\delta 2.2283}{4\sigma\pi^{\frac{1}{2}}}\right)^2. \tag{3.14}$$

Up to equation (3.13) in this analysis the only assumption we have made is that the background state is that found by the large-time asymptotics of §2 of I. In rearranging this equation to derive (3.14) we have used the assumption inherent in the analysis of the linear instabilities in I and in the weakly nonlinear analysis of §2 that δ is small. In this case this result tells us that that background state is absolutely stable if Q is less than an order-one number multiplied by δ^4, a very small parameter. When this is compared to the linear theory which predicts instabilities in the small-δ limit when Q is over about 148000 for the case of a gradient of common salt in water we see that this is a very weak result. However, it does tell us that the unstable subcritical branch of solutions found in §2 ceases to exist for sufficiently small positive values of Q.

If the Euler–Lagrange equations for the full maximization problem are derived it is found that any terms relating to the salinity stratification are absent. For small values of δ this stratification is responsible for limiting the vertical lengthscale of the instabilities to a scale much less than the thickness of the thermal layer. This results in the dissipative effects in this analysis taking place over the lengthscale of the width of the thermal layer, l, and not over the typical lengthscales determined by the salinity gradient. In the analysis of Dudis & Davis this did not weaken their results as this is also an appropriate lengthscale for the instabilities observed when a temperature gradient is heated from a sidewall. To overcome this limitation, which inhibits the usefulness of the results, we look for some additional restriction in order to improve this bound.

In experiments the instabilities are observed to be thin, almost horizontal convection cells with a vertical lengthscale of order $h^* = (\alpha\Delta T)/(-\beta\bar{S}_z)$. In the experiments of Chen *et al.* (1971) the thickest cells that they observed in their investigations into marginal stability had thickness $0.973h^*$. In the experiments of Huppert & Turner (1980) where they applied lateral heating and cooling to a vertical salinity gradient as well as melting blocks of ice submerged in a salinity gradient they found that the thickness of the layers was at most $0.89h^*$ for a case relatively close to marginal stability, but for strongly supercritical heating or cooling the thickness was more typically about $0.65h^*$. The experiments of Huppert & Josberger (1980) on melting blocks of ice in a vertical salinity gradient had a maximum layer thickness of $0.85h^*$. In none of these experiments was there a case with a layer thickness greater than h^*.

Since the instabilities tend to be almost periodic in the vertical we can adapt the previous analysis by introducing a further restriction on the allowed disturbances by only considering disturbances that are periodic in the vertical, with a period of Δz. Thus

$$(\boldsymbol{u}, T, S)(x, y, z+\Delta z, t) = (\boldsymbol{u}, T, S)(x, y, z, t). \tag{3.15}$$

We split $\boldsymbol{u}, T$ and S into parts that are independent of x and y and parts that have zero mean value when averaged over the vertical. We define

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{p}}(x, y, z, t) + \boldsymbol{u}_{\mathrm{i}}(x, t), \tag{3.16a}$$

where
$$u_i(x,t) = \overline{u(x,y,z,t)}, \tag{3.16b}$$
with similar definitions for T_i, T_p, S_i and S_p. As with the previous analysis we are again been required to choose values of λ_c and λ_d to remove the $\langle Tw \rangle$ and $\langle Sw \rangle$ terms from $\mathscr{I}$, as these terms lead to an unbounded supremum. We can then show that
$$\mathscr{I} = -\left\langle w_p u_p \frac{\partial}{\partial x} \overline{W}(x,t) \right\rangle - (\lambda_b - 1)\mathscr{R}^{\frac{1}{2}} \left\langle T_p u_p \frac{\partial}{\partial x} f(x,t) \right\rangle, \tag{3.17}$$

and so $\mathscr{I}$ is independent of the mean part of the disturbances. This enables us to use the inequalities that for any quantity A_p with period Δz in the vertical and no mean part
$$\left(\frac{2\pi}{\Delta z}\right)^2 \overline{(A_p)^2} \leqslant \overline{\left(\frac{\partial A_p}{\partial z}\right)^2} \leqslant \overline{|\nabla A_p|^2}. \tag{3.18}$$

We can use this inequality to adapt the previous analysis, and obtain the result that the fluid is stable to arbitrary disturbances of vertical periodicity Δz if
$$\sup\left\{\frac{\mathscr{I}}{\mathscr{D}}\right\} \leqslant \frac{1}{2}\left(\frac{\Delta z}{2\pi}\right)^2 \frac{(1-\tau)}{2\pi^{\frac{1}{2}}} \delta^{*3} + \frac{(1-\tau)}{2(\sigma\tau)^{\frac{1}{2}}} \mathscr{R}^{\frac{1}{2}} \left(\frac{\Delta z}{2\pi}\right)^2 \frac{\delta^*}{\pi^{\frac{1}{2}}}. \tag{3.19}$$

Again the details of this analysis are available in the document held in the *Journal of Fluid Mechanics* office and obtainable from there on request. Re-expressing this in terms of Q and δ, we find the condition that the background state is stable to arbitrary periodic disturbances of vertical period Δz if
$$Q < 4\pi \left(\frac{2\pi}{\Delta z}\right)^4 (1-\tau)^4 \left\{1 - \frac{\Delta z^2 \delta^3}{16\pi^{\frac{1}{2}}(1-\tau)^2 \pi^2}\right\}, \tag{3.20}$$
for small values of δ.

As mentioned previously, the experiments of Chen *et al.* (1971), Huppert & Josberger (1980) and Huppert & Turner (1980) always found that $\Delta z < 1$. If we use this value then we get the condition that the fluid is stable to periodic disturbances with period less that the vertical lengthscale of Chen *et al.*
$$Q < 64\pi^5 (1-\tau)^4 \{1 - O(\delta^3)\} \approx 19\,585\ (1-\tau)^4. \tag{3.21}$$

This result is much stronger than the previous result. It shows that subcritical instabilities can only exist for values of Q down to about an eighth of the value for marginal stability. In experiments with wall heating of the form of Chen *et al.*, where the wall temperature gradually rises to its final temperature, the instantaneous value of Q decays like t^{-1} after the wall temperature reaches its final level. From this we can see that even if the fluid does become unstable then the value of Q will decrease until it eventually falls below the bound given by (3.21). When this occurs periodic disturbances may still be visible; however, they will be decaying. Owing to the low diffusivity of salt the last vestiges of these instabilities may exist for some time. For experiments during which the instantaneous value of Q becomes much greater than that required for instability, observations have shown that the layer thickness will typically be of order $0.65h^*$ and not h^*. If this is used in (3.20) then, as it depends on Δz^{-4}, the bound will be improved by a factor of around 4. This could indicate an earlier onset of the decay of the periodic disturbances.

The restriction obtained on the existence of subcritical instabilities can also be used for comparison with numerical results for nonlinear disturbances with an imposed vertical periodicity (cf. Kerr 1987).

It should be noted again that there is, as yet, no theoretical justification for restricting the disturbances to a vertical periodicity of $\Delta z < 1$, or any other value. This limit for the vertical lengthscale is used because all the reported experiments satisfy this criterion. For fluids with Prandtl number $\sigma \ll 1$, and τ of a similar size, the linear theory of I predicts that the fastest growing mode has a vertical periodicity of $\Delta z \approx 1\frac{1}{2}$. For such fluids the restriction on the periodicity would have to be weakened, with a corresponding weakening of the condition (3.21).

4. Conclusion

In I the linear stability of a body of water with a vertical salinity gradient heated from a vertical sidewall was investigated for the case where the salinity gradient was strong. Although it predicted the onset of the instabilities in good agreement with experiments of Chen *et al.* (1971) and of Tanny & Tsinober (1988), it was unable to give the correct form of the observed convection cells. By its very nature the linear stability analysis predicts counter-rotating convection cells. In this paper we have investigated some of the nonlinear aspects of these instabilities in order to gain an insight into why counter-rotating cells are not observed, and to investigate some of the properties of the instabilities that would be observed.

First, the weakly nonlinear analysis showed that the bifurcation from stability was subcritical for the large range of values of σ and τ investigated, and so the form of the observed instabilities would not necessarily be similar to the form of the linear disturbances predicted at marginal stability. Although we cannot show that the subcritical branch of instabilities that bifurcates from the point of marginal linear stability is connected to any branch of solutions corresponding to instabilities observed in the experiments, we find that the perturbations to the linear stability solution show an enhancement of the convection cells that have the fluid rising near the wall and sinking away from the wall, and a corresponding reduction in the strength of the other cells. There was also a reduction in the vertical phase velocity of the instabilities. If these trends continued along the unstable subcritical branch of solutions, and this branch of solutions eventually became stable at some lower value of Q, then this would be compatible with the observed form of the solutions, with all the convection cells having the same direction of rotation.

This existence of solutions for values of Q less than the critical value predicted by the linear theory was investigated further by the use of energy stability analysis. This analysis showed that arbitrary disturbances would decay for sufficiently low values of Q. However, the lower limit for the value of Q at which instabilities may exist was a very weak bound, and not of use in practical applications. If, however, the restriction is imposed on the instabilities that they have a vertical periodicity of the order of the vertical lengthscale $\alpha\Delta T/(-\beta\bar{S}_z)$ proposed by Chen *et al.* (1971) then a much stronger result is obtained. This result shows that, for the case of common salt, the existence of subcritical solutions is limited to situations where the values of Q are greater than approximately an eighth of the value predicted by the linear theory for the onset of instability. Below this level previously established convection cells would decay. That this strengthened result produces a limit that is independent, to leading order, of δ can be anticipated from the mechanistic argument used in I to determine the form of the non-dimensional number that controls the stability of the flow. This argument also holds for finite-amplitude disturbances, and so Q would be expected to be the relevant parameter used in a criterion for the existence of subcritical disturbances.

The usefulness of this result can be seen by considering a situation where the sidewall of a salinity gradient is heated or cooled in such a way that the temperature difference between the wall is restricted to be below a certain level. Any instabilities would be limited in their vertical extent by the lengthscale proposed by Chen *et al.* However, the horizontal extent of the temperature field's penetration into the bulk of the fluid will continuously grow. The non-dimensional number Q is similar to a Rayleigh number, but instead of being proportional to a single lengthscale cubed it is proportional to the fifth power of the vertical lengthscale divided by the square of the horizontal lengthscale. From this we can see that in many real situations the instantaneous value of Q will be continually declining, and as such we can use this result to provide a limit for the existence of active convection at the sidewall due to double-diffusive processes.

The energy stability analysis can be adapted to give similar results in the case of heating a salinity gradient in a vertical slot studied by Hart (1973).

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